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The Tortuosity of Occupied Crossings of a Box in Critical Percolation

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We consider the length of an occupied crossing of a box of size $[0, n] \times [0, 3n]^{D-1}$ (in the short direction) in standard (Bernoulli) bond percolation on $\mathbb{Z}^D$ at criticality. Let $|s_n|$ be the length of the shortest such crossing. It is believed that $|s_n| \approx n^{1+c}$ in some sense for some $c > 0$. Here we show that if the correlation length $\xi(p)$ satisfies $\xi(p) \leqslant (p_c - p)^{-\nu}$ for some $\nu < 1$, then with a probability tending to 1, $|s_n| \geqslant C_1 n^{1/\nu}(\log n)^{-(1-\nu)/\nu}$. The assumption $\xi(p) \leqslant C_3(p_c - p)^{-\nu}$ with $\nu < 1$ has been rigorously established[1,2] for large D, but cannot hold[3] for $D = 2$. In the latter case, let $|l_n|$ be the length of the lowest occupied crossing of the square $[0, n]^2$. We outline a proof of $P_{p_c}(|l_n| \leqslant n^{1+c}) \leqslant n^{-\alpha}$ for some $c, \alpha > 0$. We also obtain a result about the length of optimal paths in first-passage percolation.

KEY WORDS: Critical percolation; chemical distance; tortuosity; occupied crossings of a box; lowest crossing of a square.

1. INTRODUCTION AND STATEMENT OF RESULTS

We consider standard (Bernoulli) bond percolation on $\mathbb{Z}^D$, $D \geqslant 2$, in which all bonds are independently occupied with probability p and vacant with probability $1 - p$. The corresponding probability measure on the configurations of occupied and vacant bonds is denoted by P_p. The cluster of the vertex x, $C(x)$, consists of all vertices which are connected to x by an occupied path on $\mathbb{Z}^D$. (An occupied path is a nearest-neighbor path on $\mathbb{Z}^D$, all of whose bonds are occupied.) By convention we always include x in

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$C(x)$. For any collection A of vertices, $|A|$ denotes the cardinality of A. The percolation probability is

$$\theta(p) = P_p(|C(0)| = \infty) \tag{1.1}$$

and the critical probability is

$$p_c = p_c(\mathbb{Z}^D) = \sup\{p: \theta(p) = 0\}$$

It is well known that $0 < p_c < 1$.

We still do not understand very well what the configuration of occupied and vacant bonds looks like for the system at criticality. Many questions about the critical system have been phrased in terms of the incipient infinite cluster, e.g., what is its dimension, what is the distribution of the ant in the labyrinth, how does chemical distance along the incipient infinite cluster behave, and what are the electrical resistance properties of the incipient infinite cluster (see refs. 4–10 and references cited therein)? Most of these have incomplete answers at best.

It is not even proven that $\theta(p_c) = 0$ in all dimensions (only for $D = 2$[11] or large D[1] are full proofs known), and the very definition of the incipient infinite cluster needs more work.[12] However, since it is believed that $\theta(p_c) = 0$, long occupied paths should have low probability at p_c in some sense, and therefore one expects such paths to be very tortuous. Another way to express this is that one expects the chemical distance along the incipient infinite cluster to be much larger than the Euclidean distance. If l is the chemical distance along the graph of occupied bonds between the points at Euclidean distance r, then one expects l to grow like $r^{d_{\min}}$ for some exponent $d_{\min}$. This exponent has been estimated numerically by several groups (see, for instance, refs. 9 and 10). Theorem 1 below considers the length of the shortest occupied crossing of a large rectangular box between opposite faces, assuming such a crossing exists. Presumably this length, too, behaves like $n^{d_{\min}}$ as n, the "size" of the box, become large.

We need some notation. For any two sets of vertices A and B we write $A \leftrightarrow B$ for the event that there exists an occupied path from some vertex in A to some vertex in B; $A \overset{E}{\leftrightarrow} B$ is the event that such a path exists in the region E. We set

$$B(n) = [-n, n]^D$$

and its boundary or surface is

$$\partial B(n) = \{x \in \mathbb{Z}^D: \|x\| = n\}$$

where

$$\|x\| := \max_{1 \leqslant i \leqslant D} |x_i| \qquad \text{for} \quad x = (x_1, \ldots, x_D)$$

For a path r, $|r|$ denotes its length, i.e., the number of edges in n. C_i will always stand for a strictly positive finite constant, whose value is of no significance to us. In fact the value of C_i may change from appearance to appearance.

From a simple subadditive argument (see ref. 13, Theorem 5.10) it follows that for fixed $p < p_c$ and some constants $0 < C_i < \infty$ and some function $\xi(p)$

$$C_1 n^{1-D} e^{-n/\xi(p)} \leqslant P_p(0 \leftrightarrow \partial B(n)) \leqslant C_2 n^{D-1} e^{-n/\xi(p)} \tag{1.2}$$

$\xi(p)$ is called the *correlation length.* It is a basic tenet in the theory of critical phenomena that

$$\xi(p) \approx (p_c - p)^{-\nu}, \qquad p \uparrow p_c \tag{1.3}$$

for some critical exponent ν which only depends on D. Here and in the following, $A \approx B$ means $\log A/\log B$ tends to 1 in the appropriate limit [as $p \uparrow p_c$ in (1.3)]. Hara[2] proved (1.3) for D large with $\nu = 1/2$. However, numerical data support (1.3) for all D, and with $\nu < 1$ for $D \geqslant 3$. However, it is known (see ref. 3, especially Theorem 3 and Corollary 2) that for $D = 2$, (1.3) can only hold for a $\nu > 1$; the true value of ν in $D = 2$ is believed to be[14] $\nu = 4/3$.

We write

$$F_0 = F_0(n) = \{0\} \times [0, 3n]^{D-1} \qquad \text{and} \qquad F_n = F_n(n) = \{n\} \times [0, 3n]^{D-1}$$

for the "left" and "right" faces of the box

$$D_n := [0, n] \times [0, 3n]^{D-1}$$

We denote by s_n the shortest occupied path from $F_0(n)$ to $F_n(n)$ in D_n, if such a path exists; $|s_n|$ is the length of s_n.

Theorem 1. If

$$\xi(p) \leqslant C_3 (p_c - p)^{-\nu} \qquad \text{for some } \nu < 1 \tag{1.4}$$

and some constant C_3 and p close to p_c, then there exist constants C_4 and C_5 such that

$$P_{p_c}(|s_n| \leqslant k \mid F_0(n) \leftrightarrow F_n(n)) \leqslant C_4 n^{2D-2} \exp\left(-C_5 \frac{n^{1/(1-\nu)}}{k^{\nu/(1-\nu)}}\right) \tag{1.5}$$

for all $k \geqslant n$. In particular, $\langle |s_n| \rangle_{p_c} :=$ the conditional expectation of $|s_n|$, given that there exists a crossing from F_0 to F_n in D_n (at p_c), satisfies

$$\langle |s_n| \rangle_{p_c} \geqslant C_6 n^{1/\nu} (\log n)^{-(1-\nu)/\nu} \tag{1.6}$$

for some $C_6 > 0$.

Remarks. (i) We use the box D_n instead of the simpler cube $[0, n]^D$ because of the technical reason that we know (ref. 15, Theorem 5.1 and Corollary 5.1) that

$$P_{p_c}(F_0(n) \overset{D_n}{\longleftrightarrow} F_n(n)) \geqslant C_7 > 0 \tag{1.7}$$

for some constant $C_7 > 0$. We have no proof of the analogue of (1.7) when D_n, $F_0(n)$, and $F_n(n)$ are replaced by the cube $[0, n]^D$ and its left and right faces, respectively. The result (1.7) allows us to condition on

$$\{F_0(n) \overset{D_n}{\longleftrightarrow} F_n(n)\}$$

However, the proof in Section 2 shows that if we do not insist on conditioning we still have

$$\begin{aligned} &P_{p_c}(\text{there exists an occupied crossing in } [0, n]^D \text{ from its} \\ &\quad \text{left to its right face and of length} \leqslant k) \\ &\leqslant C_4 \exp\left(-C_5 \frac{n^{n^{1/(1-\nu)}}}{k^{\nu/(1-\nu)}}\right) \end{aligned} \tag{1.8}$$

(ii) The result (1.5) also implies for any vertex v that

$$\begin{aligned} &P_{p_c}(\text{there exists an occupied path from } 0 \text{ to } v \text{ of length} \leqslant k) \\ &\leqslant C_4 \|v\|^{2D-2} \exp\left(-C_5 \frac{\|v\|^{1/(1-\nu)}}{k^{\nu/(1-\nu)}}\right), \qquad k \leqslant \|v\| \end{aligned} \tag{1.9}$$

(iii) An estimate like (1.6) can also be obtained from the following argument, which is partly nonrigorous. Let ρ_n denote the number of pivotal bonds for the event

$$\{F_0(n) \overset{D_n}{\longleftrightarrow} F_n(n)\}$$

If this event occurs, then these pivotal bonds are just the bonds which have the property that changing their state from occupied to vacant makes

$$\{F_0(n) \overset{D_n}{\longleftrightarrow} F_n(n)\}$$

fail. In the terminology of ref. 5 or ref. 7, p. 114, these are the red bonds. Every occupied crossing from $F_0(n)$ to $F_n(n)$ in D_n must go through all these pivotal edges, so that $|s_n| \geqslant \rho_n$. Now it has been argued on a nonrigorous basis that ρ_n should be of the order $n^{1/\nu}$ when $\xi(p) \approx (p_c - p)^{-\nu}$ (see ref. 7, p. 114). This would lead to a conclusion very similar to (1.6).

Unfortunately Theorem 1 is useless when $D=2$, since then (1.4) cannot hold for $\nu < 1$ (see ref. 3, Theorem 3 and Corollary 2). It is believed[14] that $\nu = 4/3$ for $D=2$. For $D=2$ we can, however, say something about the length of the lowest occupied crossing of the square $[0, n] \times [0, n]$. (The lowest crossing of a square has been used by many people; see, for instance, refs. 16 and 17; a rigorous definition is given in the Appendix of ref. 18.) Let l_n be the lowest crossing of $[0, n] \times [0, n]$ and $|l_n|$ its length.

Theorem 2. Let $\mathscr{L}_n$ be the event that there exists an occupied crossing from the left edge to the right edge in $[0, n] \times [0, n]$. Then there exist constants α, $c > 0$ and $C_1 < \infty$ such that

$$P_{p_c}(|l_n| \leqslant n^{1+c} \mid \mathscr{L}_n) \leqslant C_1 n^{-\alpha}, \qquad n \geqslant 1 \tag{1.10}$$

It is not clear that $|s_n|/|l_n| \to 0$ in probability. It is conceivable that $|s_n|$ and $|l_n|$ are of the same order, or even if $|s_n|/|l_n| \to 0$ it may be the case that

$$P_{p_c}(|s_n| \geqslant n^{-\varepsilon} |l_n| \mid \mathscr{L}_n) \to 1 \qquad \text{for every} \quad \varepsilon > 0 \tag{1.11}$$

If (1.11) is indeed the case, then $|s_n|$ and $|l_n|$ would have the same critical exponent, whenever these critical exponents exist.

The proof of (1.10) is quite involved, and we restrict ourselves to a brief outline in Section 3.

We next turn our attention to first-passage percolation on $\mathbb{Z}^D$. To each bond e of $\mathbb{Z}^D$ we associate a nonnegative random variable $X(e)$, which represents the time it takes fluid to flow through e. We assume that $\{X(e): e \in \mathbb{Z}^D\}$ are independent and all have the same distribution F. For any path r which successively traverses the edges $e_1, ..., e_n$ we define its *passage time* as

$$T(r) = \sum_{i=1}^{n} X(e_i)$$

The passage time from a vertex u to a vertex v is defined as the smallest passage time of paths from u to v; formally,

$$T(u, v) = \inf\{T(r): r \text{ a path from } u \text{ to } v\}$$

Let

$$a_{m,n} = T((m, 0,..., 0), (n, 0,...))$$
$$b_{m,n} = \inf\{T((m, 0,...,), (n, k_2,..., k_D): k_2,..., k_D \in \mathbb{Z}\}$$

$a_{m,n}$ is called a *point-to-point passage time*, and $b_{m,n}$ a *point-to-hyperplane passage time*. It is well known (ref. 19, Chapter 5, or ref. 20, Sections 2 and 3) that if

$$\int x\, dF(x) = E\{X(e)\} < \infty \tag{1.12}$$

then

$$\lim_{n \to \infty} \frac{1}{n} a_{0,n} = \lim_{n \to \infty} \frac{1}{n} b_{0,n} = \mu \quad \text{a.s.} \quad \text{and in} \quad L_1 \tag{1.13}$$

for some constant $\mu = \mu(F, D)$, the so-called *time constant*. We call r a *route* for $a_{m,n}$ (for $b_{m,n}$) if r is a path from $(m, 0,..., 0)$ to $(n, 0,..., 0)$ [to some point $(n, k_2,..., k_D)$, respectively] with $T(r) = a_{m,n}$ ($b_{m,n}$, respectively). The shortest length of such a route is $N^a_{m,n}$ or $N^b_{m,n}$: i.e.,

$$N^\theta_{m,n} = \inf\{|T(r)|: r \text{ is a route for } \theta_{m,n}\}, \qquad \theta = a \text{ or } b$$

It is believed that

$$\lim_{n \to \infty} \frac{1}{n} N^\theta_{0,n} = \lambda \quad \text{a.s.} \quad \text{and in} \quad L_1 \tag{1.14}$$

for some constant $\lambda = \lambda(F, D)$. In general this conjecture is still open, but is was shown to hold[21,22] when

$$F(0) = P\{X(e) = 0\} > p_c(\mathbb{Z}^D)$$

Some bounds for liminf and limsup of $n^{-1}N^\theta_{0,n}$ are known (ref. 19, Chapter 8, and ref. 23). Let us specialize further to distributions F for which

$$\text{supp}(F) = \{0\} \cup [1, \infty) \tag{1.15}$$

Then any $X(e)$ which is not zero must be at least one. If the conjecture $\theta(p_c) = 0$ holds, then for any F which satisfies (1.15) and

$$F(0) = p_c \tag{1.16}$$

we must have

$$a_{0,n} \to \infty \qquad \text{and} \qquad b_{0,n} \to \infty \quad \text{a.s.} \qquad (\text{as } n \to \infty)$$

We do have some bounds in the other direction. If (1.16) holds, then

$$\frac{1}{n} b_{0,n} \leqslant \frac{1}{n} a_{0,n} \to 0 \quad \text{a.s}$$

(as $n \to \infty$) (ref. 20, Theorem 6.1). If

$$P\{X(e)=0\} = 1 - P\{X(e)=1\} = p_c \tag{1.17}$$

[so that $\text{supp}(F) = \{0, 1\}$], then[24] for all $\varepsilon > 0$

$$P(b_{0,n} \leqslant n^{\varepsilon}) \geqslant P(a_{0,n} \leqslant n^{\varepsilon}) \to 1 \tag{1.18}$$

The proof of ref. 24 can be sharpened to show that under (1.17), there exists some constant $C < \infty$ such that

$$P(a_{0,n} \leqslant \exp[C(\log n)^{1/2}] \text{ for all large } n) = 1 \tag{1.19}$$

When $D = 2$ one even has

$$P(a_{0,n} \leqslant C \log n \text{ for large } n) = 1 \tag{1.20}$$

(see ref. 25, Section 3.4, for a somewhat weaker statement). We expect again that under (1.15) and (1.16) the optimal routes for $\theta_{0,n}$ must be very tortuous. They want to use as few edges e with $X(e) \geqslant 1$ as possible; in order to avoid these edges, they may have to pass through many edges f with $X(f) = 0$. Thus it is reasonable to expect that $n^{-1} N_{0,n} \to \infty$ in some sense. The next theorem confirms this under the hypotheses (1.4) and (1.17).

Theorem 3. Assume (1.4) and (1.17) hold. Then there exist constants α, $c > 0$ and $0 < C_1, C_2 < \infty$ such that

$$P(N^{\theta}_{0,n} \leqslant n^{1+c}) \leqslant C_1 e^{-C_2 n^{\alpha}}, \qquad n \geqslant 1 \quad \text{and} \quad \theta = a \text{ or } b \tag{1.21}$$

2. PROOFS OF THEOREM 1 AND 3

Proof of Theorem 1. We can choose the configuration for given $p < p_c$ by first choosing the configuration for p_c and then doing a second random experiment in which every occupied edge at p_c remains occupied (becomes vacant) with probability $p/p_c[(p_c - p)/p$, respectively]. From this one obtains [compare ref. (15), inequality (4.2)]

$$\begin{aligned} &P_p(F_0 \overset{D_n}{\longleftrightarrow} F_n, |s_n| = l) \\ &\quad \geqslant P\{\text{shortest occupied crossing at } p_c \text{ has length } l \text{ and stays occupied at } p\} \\ &\quad \geqslant \left(\frac{p}{p_c}\right)^l P_{p_c}(F_0 \overset{D_n}{\longleftrightarrow} F_n, |s_n| = l) \end{aligned}$$

Hence by summing over $l \leqslant k$, for any k, and $p < p_c$

$$P_p(F_0 \overset{D_n}{\longleftrightarrow} F_n) \geqslant \left(\frac{p}{p_c}\right)^k P_{p_c}(F_0 \overset{D_n}{\longleftrightarrow} F_n, |s_n| \leqslant k) \tag{2.1}$$

Since clearly

$$\{F_0 \overset{D_n}{\longleftrightarrow} F_n\} \subset \bigcup_{v \in F_0} \{v \leftrightarrow v + \partial B(n)\}$$

(2.1), (1.2), and (1.7) imply, for any choice of k,

$$\begin{aligned} P_{p_c}(|s_n| \leqslant k \mid F_0 \overset{D_n}{\longleftrightarrow} F_n) &\\ \leqslant \left(\frac{1}{C_7}\right)\left(\frac{p_c}{p}\right)^k \sum_{v_0 \in F_0} C_2 n^{D-1} e^{-n/\xi(p)} &\\ \leqslant C_8 n^{2D-2} \exp\left\{k \frac{p_c - p}{p} - \frac{n}{\xi(p)}\right\} & \end{aligned}$$

If (1.4) holds, and we restrict ourselves to $\frac{1}{2} p_c \leqslant p < p_c$, we have

$$\begin{aligned} P_{p_c}(|s_n| \leqslant k \mid F_0 \overset{D_n}{\longleftrightarrow} F_n) &\\ \leqslant C_4 n^{2D-2} \exp[C_9 k(p_c - p) - C_3^{-1} n(p_c - p)^{\nu}] & \end{aligned}$$

(1.5) follows by taking $p_c - p = (n\nu/C_3 C_9 k)^{1/(1-\nu)} \wedge (\frac{1}{2} p_c)$ (which minimizes the exponent in the last display). (1.6) is immediate from (1.5). ∎

Proof of Theorem 3. Fix $0 < \varepsilon < 1 - \nu$. Then for $\theta = a, b$

$$P(N^{\theta}_{0,n} \leqslant k) \leqslant P(N^{\theta}_{0,n} \leqslant k, \theta_{0,n} < n^{\varepsilon}) + P(\theta_{0,n} \geqslant n^{\varepsilon}) \tag{2.2}$$

By the result of ref. 24 (see in particular the hypothesis H_b, which is eventually proved), we have for large n

$$P(b_{0,n} \geqslant n^{\varepsilon}) \leqslant P(a_{0,n} \geqslant n^{\varepsilon}) \leqslant C_6 e^{-C_7 n^{\varepsilon/2}} \tag{2.3}$$

for some constants $C_6 = C_6(\varepsilon)$ and $C_7 = C_7(\varepsilon) > 0$. Furthermore, if r is a route for $\theta_{0,n}$, then r contains exactly $\theta_{0,n}$ edges e with $X(e) = 1$. Thus, r can be broken up into $\gamma := \theta_{0,n} + 1$ subpaths $r_1, ..., r_{\gamma}$, each of which contains only edges e with $X(e) = 0$, with the exception of its last edge, which may have a corresponding X equal to 1. For any path s, set

$$d(s) = \|(\text{endpoint of } s) - (\text{initial point of } s)\|$$

Since the distance between the endpoint and initial point of r is at least n (because r is a route for $\theta_{0,n}$), we must have $d(s_i) \geqslant n/\gamma = n/(\theta_{0,n}+1)$ for at least one $i \leqslant \gamma$. Moreover, if $|r|$ = the length of r is $\leqslant k$, then r, and hence r_i, is contained in $[-k, k]^D$. Of course also $|r_i| \leqslant |r|$. Therefore,

$$\{N^{\theta}_{0,n} \leqslant k, \theta_{0,n} < n^{\varepsilon}\}$$

$$\subset \bigcup_{x, y \in [-k,k]^D, \|x-y\| \geqslant n^{1-\varepsilon}-2} \{\exists \text{ a path } s \text{ from } x \text{ to } y \text{ with } |s| \leqslant k \text{ and with } X(e)=0 \text{ for all edges } e \text{ in } s\}$$

It follows from this and (1.9) that

$$P(N^{\theta}_{0,n} \leqslant k, \theta_{0,n} < n^{\varepsilon})$$

$$\leqslant \sum_{x, y \in [-k,k]^D, \|x-y\| \geqslant n^{1-\varepsilon}-2} P_{P_c}(\exists \text{ an occupied path } s \text{ from } x \text{ to } y \text{ with } |s| \leqslant k)$$

$$\leqslant (2k+1)^D \sum_{n^{1-\varepsilon}-2 \leqslant \|y\| \leqslant 2Dk} C_4 \|y\|^{2D-2} \exp\left(-C_5 \frac{\|y\|^{1/(1-\nu)}}{k^{\nu/(1-\nu)}}\right)$$

$$\leqslant C_8 k^{4D-2} \exp\left(-C_5 \frac{n^{(1-\varepsilon)/(1-\nu)}}{k^{\nu/(1-\nu)}}\right) \tag{2.4}$$

Now choose

$$0 < \varepsilon < \frac{2(1-\nu)}{3-\nu}, \qquad \beta = \frac{1-\nu}{2}\varepsilon \tag{2.5}$$

Then for

$$k \leqslant n^{(1-\varepsilon-\beta)/\nu}$$

the right-hand side of (2.4) is at most

$$C_9 \exp\left(-\frac{C_5}{2} n^{\beta/(1-\nu)}\right) = C_9 \exp\left(-\frac{C_5}{2} n^{\varepsilon/2}\right)$$

for large n. Thus (1.21) follows from (2.3) and (2.4) with

$$1 + c = \frac{1-\varepsilon-\beta}{\nu} > 1$$

3. OUTLINE OF PROOF OF THEOREM 2

Denote expectation with respect to P_{p_c} by E, and let

$$S_n = \left[\frac{n}{2} - \frac{n}{64}, \frac{n}{2} + \frac{n}{64}\right]^2 \tag{3.1}$$

Finally, let $|l_n \cap S_n|$ denote the number of edges of l_n in S_n. Then the three principal steps in the proof of Theorem 2 are the following:

(a) $E\{|l_n \cap S_n| \mid \mathscr{L}_n\} \geqslant n^{1+2c}$ for some $c>0$ and n large.

(b) $E\{|l_n \cap S_n|^2 \mid \mathscr{L}_n\} \leqslant C_2[E\{|l_n \cap S_n| \mid \mathscr{L}_n\}]^2$ for some constant $C_2 < \infty$.

(c) Deduction of (1.10) from (a) and (b).

Heuristically, (a) seems to us the most important step. Its proof relies heavily on the methods of refs. 3 and 26. To prove (a), we begin with the following observation. We say that a vertex $v = (v_1, v_2)$ is the *last vertex of* l_n *on the vertical line* $\{x = v_1\}$ if l_n passes through v, but no longer intersects this line after v. Thus, if $l_n = (e_1,..., e_\sigma)$ with e_1 (e_σ) having one endpoint on the left (right) edge of $[0, n] \times [0, n]$, then v is the common endpoint of some e_i and e_{i+1}, and $e_{i+1}, e_{i+2},..., e_\sigma$ have no other endpoints on $\{x = v_1\}$. Each vertical line $\{x = v_1\}$ with $0 \leqslant v_1 \leqslant n$ has exactly one last vertex of l_n and it is easy to see that the expected number of last vertices of l_n in S_n is at least $C_3 n$ for some $C_3 > 0$. The difficult part of (a) is to show that for any v in the square S_n and some $c > 0$

$$\begin{aligned} &P_{p_c}(l_n \text{ passes through } v \mid \mathscr{L}_n) \\ &\qquad \geqslant n^{2c} P_{p_c}(v \text{ is the last vertex of } l_n \text{ on } \{x = v_1\} \mid \mathscr{L}_n) \end{aligned} \tag{3.2}$$

Once one has (3.2), (a) follows, since then

$$\begin{aligned} &E\{|l_n \cap S_n| \mid \mathscr{L}_n\} \\ &\qquad \geqslant n^{2c} \sum_{v \in S_n} P_{p_c}(v \text{ is the last vertex of } l_n \text{ on } \{x = v_1\} \mid \mathscr{L}_n) \geqslant C_3 n^{1+2c} \end{aligned}$$

The inequality (3.2) itself is proven by a method similar to the one in ref. 26. Consider the annulus

$$A_k(v) = v + ([-2^k, 2^k]^2 \setminus (2^{k-1}, 2^{k-1})^2)$$

around v. If $A_k(v) \subset [0, n]^2$ and if there exists a vacant "dual half circuit in the right half of $A_k(v)$" (i.e., a vacant path on $\mathbb{Z}^2 + (1/2, 1/2)$ inside $A_k(v) \cap \{(x, y): x \geqslant v_1 - 1/2\}$, which connects $\{v_1\} \times [v_2 + 2^{k-1}, v_2 + 2^k]$

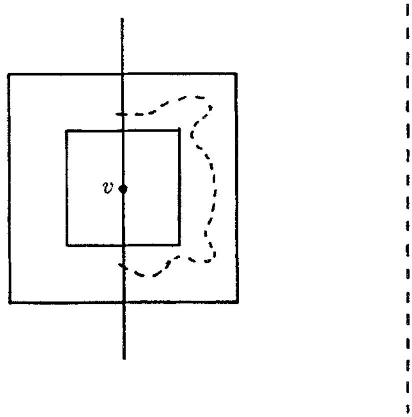

Fig. 1. $A_k(v)$ is the annulus between the inner and outer squares. If the dashed "half circuit" in $A_k(v)$ is vacant, then it prevents the occurrence of $\{v$ is the last v vertex of l_n on $\{x=v_1\}\}$. In order to connect v to the right edge of $[0,n]^2$ (the dashed vertical line on the right) without hitting the dashed half circuit one has to reenter $\{x<v_1\}$.

to $\{v_1\}\times[v_2-2^k, v_2-2^{k-1}]$ (see Fig. 1)), then v cannot be the last vertex of l_n on $\{x=v_1\}$. As Fig. 1 indicates, any occupied path from v to the right edge of $[0,n]^2$ [which lies outside $A_k(v)$] must enter the region $\{x<v_1\}$ again in order to avoid the vacant half circuit. There are at least $C_4 \log n$ values of k with $A_k(v)\subset[0,n]^2$, and for each such k there is a conditional probability at least $C_5>0$ of having a vacant half circuit. This yields (after considerable technical work) that

$$\begin{aligned}&P_{p_c}(v \text{ is the last vertex of } l_n \mid \mathscr{L}_n)\\&\qquad\leqslant (1-C_5)^{C_4\log n}\, P_{p_c}(l_n \text{ passes through } v \mid \mathscr{L}_n)\end{aligned}$$

This is of course the same as (3.2).

For (b) we use arguments of ref. 3, pp. 147, 148. We first prove that

$$P_{p_c}(l_n \text{ passes through } v \mid \mathscr{L}_n)$$

is of the same order of magnitude as

$$P_{p_c}(v \text{ is connected by two edge disjoint occupied paths to the boundary of } [0,n]^2 \text{ and one of the points } v+(\pm 1/2, \pm 1/2) \text{ is connected by a dual vacant path to the boundary of } [0,n]^2)$$

This then leads, for two points v and w in S_n with $\|v-w\| \geqslant k$, to the estimate

$$\begin{aligned} &P_{p_c}(l_n \text{ passes through } v \text{ and } w \mid \mathscr{L}_n) \\ &\quad \leqslant P_{p_c}\left(l_n \text{ passes through } v \mid \mathscr{L}_n\right) \\ &\qquad \times P_{p_c}\left(l_k \text{ passes through } \left(\left[-\frac{k}{2}\right], \left[\frac{k}{2}\right]\right) \middle| \mathscr{L}_k\right) \end{aligned} \tag{3.3}$$

The sum over all v and w in S_n of the left-hand side in (3.3) can then be shown to be only of order

$$[E\{|l_n \cap S_n| \mid \mathscr{L}_n\}]^2$$

from which (b) follows.

It is a simple consequence of Schwarz' inequality (see ref. 27, Exercise 1.3.5) that (a) and (b) together imply

$$\begin{aligned} &P_{p_c}\left(|l_n \cap S_n| \geqslant \frac{1}{2} n^{1+2c} \mid \mathscr{L}_n\right) \\ &\quad \geqslant P_{p_c}\left(|l_n \cap S_n| \geqslant \frac{1}{2} E\{|l_n \cap S_n| \mid \mathscr{L}_n\} \mid \mathscr{L}_n\right) \\ &\quad \geqslant \frac{1}{4} \frac{[E\{|l_n \cap S_n| \mid \mathscr{L}_n\}]^2}{E\{|l_n \cap S_n|^2 \mid \mathscr{L}_n\}} \geqslant \frac{1}{4C_2} \end{aligned} \tag{3.4}$$

The right-hand side of (3.4) is bounded away from 0, but does not yet tend to 1. To obtain the full result (1.10), one needs to find a number of disjoint squares, to each one of which one can apply (3.4). Step (c) consists of finding such squares (at random locations) of size $n^{1-\gamma}$ for a suitable $\gamma > 0$.

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